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SOME CHARACTERIZATIONS OF n-DIMENSIONAL F-SPACES

BY
M. J. CANFELL(1)

Abstract. In this paper we obtain characterizations of an n-dimensional F-space in terms of the rings of continuous real-valued and complex-valued functions defined on the space. Motivation for these results is the work of Gillman and Henriksen on U-spaces (F-spaces of dimension 0) and T-spaces (F-spaces of dimension 0 or 1).

1. **Introduction.** Throughout, X denotes a completely regular (Hausdorff) space, $C(X)$ the ring of continuous real-valued functions on X, and $C^*(X)$ the subring of $C(X)$ consisting of the bounded functions in $C(X)$.

By definition, X is an F-space if $C(X)$ has the property that finitely generated ideals in $C(X)$ are principal [5], [6]. Our main concern here is to define a condition on commutative rings with identity in such a way that X is an n-dimensional F-space if and only if $C(X)$ satisfies this condition. The condition we select, called H_n, corresponds to condition T of [4] when $n=1$. In Theorem 3, we prove that X is an n-dimensional F-space if and only if $C(X)$ satisfies condition H_n. Characterizations of topological dimension alone in terms of $C(X)$ have been given in [2] and [6, Theorem 16.35].

In Theorems 3 and 4 we give characterizations of F-spaces and n-dimensional F-spaces in terms of the rings of continuous complex-valued functions defined on them. These characterizations are analogous to those in terms of $C(X)$ and are of interest in connection with sup-norm algebras of complex continuous functions [8], and alignable complex Banach lattices [1].

For $f \in C(X)$ we define $Z(f)=\{x \in X : f(x)=0\}$ (the zero-set of f), $P(f)=\{x \in X : f(x)>0\}$ and $N(f)=\{x \in X : f(x)<0\}$. For the elementary properties of zero-sets the reader is referred to [6].

We use the modification of covering dimension involving basic covers given in [6, p. 243]. By a slight modification of Definition 4 of [3], we obtain the following characterization of dimension.

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LEMMA 1. $\dim X \leqq n$ *if and only if given* $n+1$ *disjoint pairs* C_i, C_i', $i=1,\ldots,n+1$, *of zero-sets of* X, *there exist functions* $k_i \in C(X)$ *such that* $k_i(C_i)=\{1\}$, $k_i(C_i')=\{-1\}$, $-1 \leqq k_i \leqq 1$, *and* $\bigcap_{i=1}^{n+1} Z(k_i)=\varnothing$.

Proof. *Necessity.* If C_i and C_i' are disjoint zero-sets, we can choose $f_i \in C(X)$ such that $f_i(C_i)=\{1\}$, $f_i(C_i')=\{-1\}$ and $-1 \leqq f_i \leqq 1$. Let $I^{n+1}=[-1,1]^{n+1}$ and let S^n denote the surface of I^{n+1}. Then $f=(f_1,\ldots,f_{n+1})$ is a continuous mapping of X into I^{n+1}. Since $\dim X \leqq n$, we can, by Definition 3 of [3], choose $k=(k_1,\ldots,k_{n+1})\colon X \to S^n$ such that $k(x)=f(x)$ whenever $f(x) \in S^n$. Then the functions k_i, $i=1,\ldots,n+1$, satisfy the required conditions.

Sufficiency. If functions k_i exist as stated, then C_i and C_i' are separated in $X-Z(k_i)$ and $\bigcap_{i=1}^{n+1} Z(k_i)=\varnothing$. By Definition 4 of [3], $\dim X \leqq n$.

We now recall some properties of F-rings and Hermite rings. In the following S will denote a commutative ring with identity. The ideal of S generated by n elements $a_1,\ldots,a_n$ will be denoted by $a_1S+\cdots+a_nS$. A commutative ring S with identity is called an F-ring if every finitely generated ideal of S is principal. Thus X is an F-space if and only if $C(X)$ is an F-ring.

We take the following characterization of Hermite rings [4, Lemma 4].

LEMMA 2. *A commutative ring* S *with identity is a Hermite ring if and only if it satisfies the conditions*:

(i) S *is an* F-*ring.*

(ii) *Whenever* $a_1, a_2, d \in S$ *and* $a_1S+a_2S=dS$, *there exist* $b_1, b_2 \in S$ *such that* $a_1=b_1d$, $a_2=b_2d$ *and* $b_1S+b_2S=S$.

A completely regular space X such that $C(X)$ is a Hermite ring is called a T-space. Alternative characterizations of T-spaces are given in [5, Theorem 3.2]. We will see later that X is a T-space if and only if X is an F-space and $\dim X \leqq 1$.

2. n-dimensional F-spaces.

DEFINITION. Let n be a nonnegative integer. A commutative ring S with identity is said to be an H_n-ring, or to satisfy the condition H_n, if

(i) S is an F-ring.

(ii) Whenever $a_1,\ldots,a_{n+1}$, $d \in S$ and $a_1S+\cdots+a_{n+1}S=dS$, there exist $b_1,\ldots,b_{n+1} \in S$ such that $a_1=b_1d,\ldots,a_{n+1}=b_{n+1}d$ and $b_1S+\cdots+b_{n+1}S=S$.

Thus S is an H_1-ring if and only if it is a Hermite ring, and S is an H_0-ring if and only if it is an F-ring in which generators of principal ideals are unique (up to associates).

THEOREM 3. *For every completely regular space* X, *the following statements are equivalent*:

(a) X *is an* F-*space and* $\dim X \leqq n$.

(b) $C(X)$ *is an* H_n-*ring.*

(c) $C^*(X)$ *is an* H_n-*ring.*

(d) *For all* $f_1,\ldots,f_{n+1} \in C(X)$, *there exist* $k_1,\ldots,k_{n+1} \in C(X)$ *such that* $f_1=k_1|f_1|,\ldots,f_{n+1}=k_{n+1}|f_{n+1}|$ *and* $k_1C(X)+\cdots+k_{n+1}C(X)=C(X)$.

Proof. (a) $\Rightarrow$ (d). Suppose $f_1,\dots,f_{n+1}\in C(X)$. Since X is an F-space, $P(f_i)$ and $N(f_i)$ are contained in disjoint zero-sets. By Lemma 1, there exist functions k_i such that $k_i(P(f_i))=\{1\}$, $k_i(N(f_i))=\{-1\}$, and $\bigcap_{i=1}^{n+1} Z(k_i)=\varnothing$. Hence $f_i=k_i|f_i|$, $i=1,\dots,n+1$, and $k_1C(X)+\cdots+k_{n+1}C(X)=C(X)$.

(d) $\Rightarrow$ (b). The hypothesis implies that X is an F-space and hence that $C(X)$ is an F-ring. Suppose that $f_1C(X)+\cdots+f_{n+1}C(X)=hC(X)$. There exist $g_1',\dots,g_{n+1}'\in C(X)$ and $s_1,\dots,s_{n+1}\in C(X)$ such that $f_1=g_1'h,\dots,f_{n+1}=g_{n+1}'h$ and $h=s_1f_1+\cdots+s_{n+1}f_{n+1}=s_1g_1'h+\cdots+s_{n+1}g_{n+1}'h$. Put $q=1-s_1g_1'-\cdots-s_{n+1}g_{n+1}'$.

Then $hq=0$ and for any elements $t_i\in C(X)$ we have $(g_i'+t_iq)h=f_i$. We will choose the t_i so that the elements $g_i'+t_iq$ generate $C(X)$. Since X is an F-space, there exists $p\in C(X)$ such that $pq=|q|$. By hypothesis, there exist $m_i\in C(X)$ such that $g_i'=m_i|g_i'|$, $i=1,\dots,n+1$, and $\bigcap_{i=1}^{n+1} Z(m_i)=\varnothing$. Let $t_i=pm_i$ and let $g_i=g_i'+t_iq$. Then for each $x\in X$, we have $g_i(x)\neq 0$ for some i. To see this, suppose first that $g_i'(x)\neq 0$ for some i. Now $(t_iq)(x)=p(x)m_i(x)q(x)=m_i(x)|q(x)|$ has the same sign (or argument) as $g_i'(x)$ so that $g_i(x)\neq 0$. On the other hand, if $g_i'(x)=0$ for all i, then $q(x)=1$, $p(x)=1$, and $g_i(x)=t_i(x)=m_i(x)$. Since $\bigcap_{i=1}^{n+1} Z(m_i)=\varnothing$, then $g_i(x)\neq 0$ for some i. Hence $g_1C(X)+\cdots+g_{n+1}C(X)=C(X)$.

(b) $\Rightarrow$ (a). By hypothesis, $C(X)$ is an F-ring and hence X is an F-space. Suppose that C_i, C_i', $i=1,\dots,n+1$, are $n+1$ disjoint pairs of zero-sets. Choose $f_i\in C(X)$ such that $f_i(C_i)=\{1\}$, $f_i(C_i')=\{-1\}$, for $i=1,\dots,n+1$, and let $h=|f_1|+\cdots+|f_{n+1}|$. Since X is an F-space, $f_1C(X)+\cdots+f_{n+1}C(X)=hC(X)$. By hypothesis, there exist $g_i\in C(X)$ such that $f_i=g_ih$ and $g_1C(X)+\cdots+g_{n+1}C(X)=C(X)$. Thus $\bigcap_{i=1}^{n+1} Z(g_i)=\varnothing$. Now $P(f_i)\subset P(g_i)$, $N(f_i)\subset N(g_i)$ for $i=1,\dots,n+1$. Also $|g_i(x)|\leqq 1$ for $f_i(x)\neq 0$ and we can arrange that $|g_i(x)|\leqq 1$ everywhere (take $g_i'(x)=g_i(x)$ if $|g_i(x)|\leqq 1$ and $g_i'(x)=g_i(x)/|g_i(x)|$ if $|g_i(x)|\geqq 1$). Since $P(g_i)$ and $N(g_i)$ are completely separated, we can choose s_i so that $s_i(P(g_i))=\{1\}$ and $s_i(N(g_i))=\{0\}$.

Let $m_i\in C(X)$ satisfy $f_i=m_i|f_i|$, $-1\leqq m_i\leqq 1$. Define $k_i=s_i\max\{m_i,g_i\}+(1-s_i)\min\{m_i,g_i\}$. Then $f_i=k_i|f_i|$ and $Z(k_i)\subset Z(g_i)$. Hence $\bigcap_{i=1}^{n+1} Z(k_i)=\varnothing$. Since $k_i(C_i)=\{1\}$ and $k_i(C_i')=\{-1\}$ we have $\dim X\leqq n$ by Lemma 1.

(b) $\Leftrightarrow$ (c). $C^*(X)$ is isomorphic to $C(\beta X)$ where βX is the Stone-Čech compactification of X. Since $\dim X=\dim \beta X$ [6, p. 245] and X is an F-space if and only if βX is an F-space, the result follows from (a) $\Leftrightarrow$ (b) above.

Example. *βR^n-R^n is an n-dimensional F-space.*

That βR^n-R^n is an F-space follows from Theorem 14.27 of [6], and it is shown in [7] that $\dim(\beta R^n-R^n)=n$.

As a simple consequence, we have an example of an F-ring which is not a Hermite ring (the first example of this was given in [5]). βR^2-R^2 is an F-space which is not a T-space, hence $C(\beta R^2-R^2)$ is an F-ring which is not a Hermite ring.

3. **Continuous complex functions on F-spaces.** We turn now to the problem of characterizing F-spaces in terms of the ring $C_C(X)$ of all continuous complex-

valued functions on X. We also consider $C_C^*(X)$, the subring of $C_C(X)$ consisting of the bounded functions in $C_C(X)$.

Since $Z(f)=Z(|f|)$, the family of zero-sets of $C_C(X)$ is the same as the family of zero-sets of $C(X)$.

An ideal I of $C_C(X)$ is said to be *selfadjoint* if and only if $f\in I \Rightarrow \bar{f}\in I$, where $\bar{f}$ is the complex conjugate of f.

THEOREM 4. *The following conditions are equivalent*:

(a) *X is an F-space.*

(b) *$C_C(X)$ is an F-ring.*

(c) *$C_C^*(X)$ is an F-ring.*

(d) *Each ideal I of $C_C(X)$ is selfadjoint.*

(e) *For all $f, g\in C_C(X)$, $fC_C(X)+gC_C(X)=(|f|+|g|)C_C(X)$.*

(f) *Given a zero-set Z of X, every function $\theta\in C_C^*(X-Z)$ has a continuous extension $h\in C_C^*(X)$.*

(g) *Given $f\in C_C(X)$, there exist $k_1, k_2\in C_C(X)$ such that $f=k_1|f|$ and $|f|=k_2 f$.*

Proof. (g) $\Rightarrow$ (d). Let $f\in I$. There exist $k_1, k_2\in C_C(X)$ such that $\bar{f}=k_1|\bar{f}|=k_1|f|$ and $|f|=k_2 f$. Hence $\bar{f}=k_1k_2f$ so that $\bar{f}\in I$.

(d) $\Rightarrow$ (a). Let $f\in C(X)$. Then $f-i|f|\in C_C(X)$ and by hypothesis its complex conjugate $f+i|f|$ is in the principal ideal generated by $f-i|f|$. There exists $h\in C_C(X)$ such that $f+i|f|=h(f-i|f|)$. On multiplying both sides by $f-i|f|$ we have

$$f^2+|f|^2 = h(f^2-2i|f|f-|f|^2),$$

and on simplifying and equating real parts, we get

$$|f|^2 = f^2 = I(h)f|f|.$$

It follows that $f=I(h)|f|$ so that X is an F-space.

The rest of the proof is a routine modification of the proofs in Theorem 14.25 of [6]. For example, (a) $\Rightarrow$ (f) since the real and imaginary parts of θ can be extended over X.

Although $C_C(X)$ is an F-ring if and only if $C(X)$ is an F-ring, the situation is slightly different for H_n-rings.

THEOREM 5. *The following conditions are equivalent*:

(a) *$C_C(X)$ is an H_n-ring.*

(b) *$C(X)$ is an H_{2n+1}-ring.*

(c) *X is an F-space and* $\dim X\leqq 2n+1$.

(d) *For all $f_1,\ldots,f_{n+1}\in C_C(X)$, there exist $k_1,\ldots,k_{n+1}\in C_C(X)$ such that $f_1=k_1|f_1|,\ldots,f_{n+1}=k_{n+1}|f_{n+1}|$ and $k_1C_C(X)+\cdots+k_{n+1}C_C(X)=C_C(X)$.*

Proof. (a) $\Rightarrow$ (b). First we observe that if X is an F-space and $f_1, f_2\in C(X)$, then $f_1C(X)+f_2C(X)=(f_1^2+f_2^2)^{1/2}C(X)$. In fact, since $(f_1^2+f_2^2)^{1/2}\leqq|f_1|+|f_2|$

$\leqq 2(f_1^2+f_2^2)^{1/2}$, then it follows from Theorem 14.25(6) of [6], that $|f_1|+|f_2|$ and $(f_1^2+f_2^2)^{1/2}$ are multiples of each other. Similarly if $f_1, f_2 \in C_C(X)$, then $f_1C_C(X) +f_2C_C(X)=(|f_1|^2+|f_2|^2)^{1/2}C_C(X)$.

Now suppose that $f_1, \ldots, f_{2n+2}, d \in C(X)$ and $f_1C(X)+\cdots+f_{2n+2}C(X) =dC(X)$. Let $h=(f_1^2+\cdots+f_{2n+2}^2)^{1/2}$. By hypothesis and Theorem 4, X is an F-space and, by the preceding remarks, $dC(X)=hC(X)$. Let $g_i=f_{2i-1}+if_{2i}$, $i=1, \ldots, n+1$. Again by the preceding remarks, $g_1C_C(X)+\cdots+g_{n+1}C_C(X) =(|g_1|^2+\cdots+|g_{n+1}|^2)^{1/2}C_C(X)=hC_C(X)$. Therefore $g_1C_C(X)+\cdots+g_{n+1}C_C(X) =dC_C(X)$. By hypothesis, there exist elements $s_{2i-1}+is_{2i} \in C_C(X)$ which generate $C_C(X)$ and which satisfy $g_i=f_{2i-1}+if_{2i}=(s_{2i-1}+is_{2i})d$. Thus $f_i=s_id$, $i=1, \ldots, 2n+2$ and $\bigcap_{i=1}^{2n+2} Z(s_i)=\varnothing$, i.e., $s_1C(X)+\cdots+s_{2n+2}C(X)=C(X)$.

(b) $\Rightarrow$ (c). This has been shown in Theorem 2.

(c) $\Rightarrow$ (d). Let $f_1, \ldots, f_{n+1} \in C_C(X)$. By Theorem 4, there exist $k_1', \ldots, k_{n+1}' \in C_C(X)$ such that $f_1=k_1'|f_1|, \ldots, f_{n+1}=k_{n+1}'|f_{n+1}|$. If $f_i(x)\neq 0$, then $|k_i'(x)|=1$, and we may assume that $|k_i'(x)|\leqq 1$ for $x \in X$, $i=1, \ldots, n+1$.

Let D be the closed unit disc in the complex plane and D_1 its surface; that is, $D=\{z \in C : |z|\leqq 1\}$ and $D_1=\{z \in C : |z|=1\}$. Then $k'=(k_1', \ldots, k_{n+1}')$ is a continuous mapping of X into $D^{n+1}\subset R^{2n+2}$. Since dim $X\leqq 2n+1$, we may, as in Definition 3 of [3], choose $k=(k_1, \ldots, k_{n+1})\colon X \to D_1^{n+1}$ such that $k(x)=k'(x)$ whenever $k'(x) \in D_1^{n+1}$. Thus $f_i=k_i|f_i|$, $i=1, \ldots, n+1$, and $\bigcap_{i=1}^{n+1} Z(k_i)=\varnothing$.

(d) $\Rightarrow$ (a). The proof is identical with (d) $\Rightarrow$ (b) of Theorem 3.

COROLLARY. *X is a T-space if and only if given $f \in C_C(X)$, there exists $k \in C_C(X)$ such that $f=k|f|$ and $Z(k)=\varnothing$.*

Proof. This is (d) $\Leftrightarrow$ (b) above with $n=0$ but we give a simple direct proof. If X is an F-space and $f=f_1+if_2$, $k=k_1+ik_2$, then $f=k|f|$ and $Z(k)=\varnothing$ if and only if $f_1=k_1(f_1^2+f_2^2)^{1/2}$, $f_2=k_2(f_1^2+f_2^2)^{1/2}$ and $Z(k_1)\cap Z(k_2)=\varnothing$. Since $f_1C(X) +f_2C(X)=(f_1^2+f_2^2)^{1/2}C(X)$, then (b) $\Rightarrow$ (d) is immediate, while (d) $\Rightarrow$ (b) follows from Lemma 4 of [4].

As the example $X=\beta R^2-R^2$ shows, $C_C(X)$ may be a Hermite ring while $C(X)$ is not a Hermite ring.

4. ***U*-spaces and *T*-spaces.** An element u of $C(X)$ (or $C_C(X)$) is said to be *unitary* if $|u(x)|=1$ for all $x \in X$. If $f=v|f|$ and $Z(v)=\varnothing$, then $u=v/|v|$ is unitary, and since $|f|=|v|\,|f|$, we have $f=v|f|=u|v|\,|f|=u|f|$.

From Theorem 3 and the corollary to Theorem 5, we have the following characterization.

LEMMA 6. *X is a U-space (respectively T-space) if and only if for each $f \in C(X)$ (respectively $C_C(X)$), there exists a unitary element u of $C(X)$, (respectively $C_C(X)$) such that $f=u|f|$.*

Finally we given an unpublished result of Bonsall in which T-spaces are characterized in terms of linear operators on the complex vector space $C_C(X)$.

A *rotation* on $C_C(X)$ is a linear operator D mapping $C_C(X)$ onto $C_C(X)$ such that $|Df|=|f|$ for all $f \in C_C(X)$. $C_C(X)$ is said to be *alignable* if and only if given $f_0 \in C_C(X)$ there exists a rotation D on $C_C(X)$ such that $D|f_0|=f_0$.

Alignable spaces were considered in [1].

THEOREM 7. *X is a T-space if and only if $C_C(X)$ is alignable.*

Proof. If $u \in C_C(X)$ is a unitary element for which $f_0=u|f_0|$, then clearly the operation of multiplication by u is a rotation on $C_C(X)$ with the required property.

Conversely, suppose that D is a rotation on $C_C(X)$ for which $D|f_0|=f_0$. We show that $D1$ is unitary and that D is the operation of multiplication by $D1$. Given $x \in X$, let Ψ_x and Φ_x denote the linear functionals on $C_C(X)$ defined by $\Psi_x(f)=f(x)$ and $\Phi_x(f)=(Df)(x)$. Then $|\Psi_x(f)|=|\Phi_x(f)|$ for each $f \in C_C(X)$. Hence Ψ_x and Φ_x have the same null space and therefore differ only by a scalar factor. Thus $\Phi_x=\lambda_x\Psi_x$ for some $\lambda_x \in C$ with $|\lambda_x|=1$. Now $(Df)(x)=\Phi_x(f)=\lambda_x\Psi_x(f)=\lambda_x f(x)$. In particular, $(D1)(x)=\lambda_x$ so that $(Df)(x)=((D1)(x))f(x)$. This holds for all $x \in X$, so that $Df=(D1)f$. Finally, for each $x \in X$, $|(D1)(x)|=|\lambda_x|=1$ so that $D1$ is unitary and $(D1)|f_0|=D|f_0|=f_0$.

REFERENCES

1. F. F. Bonsall and B. J. Tomuik, *The semi-algebra generated by a compact linear operator*, Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 177–196. MR **32** #1557.

2. M. J. Canfell, *Uniqueness of generators of principal ideals in rings of continuous functions*, Proc. Amer. Math. Soc. **26** (1970), 571–573.

3. J. R. Gard and R. D. Johnson, *Four-dimension equivalences*, Canad. J. Math. **20** (1968), 48–50. MR **36** #5913.

4. L. Gillman and M. Henriksen, *Some remarks about elementary divisor rings*, Trans. Amer. Math. Soc. **82** (1956), 362–365. MR **18**, 9.

5. ———, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. **82** (1956), 366–391. MR **18**, 9.

6. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

7. M. Jerison, *Rings of germs of continuous functions*, Proc. Conf. Functional Analysis (Irvine, Calif., 1966), Academic Press, London; Thompson Book, Washington, D.C., 1967. MR **36** #4342.

8. M. L. Weiss, *Some separation properties in sup-norm algebras of continuous functions*, Proc. Internat. Sympos. Function Algebras (Tulane University, 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 93–97. MR **33** #1756.

UNIVERSITY OF NEW ENGLAND,
ARMIDALE, AUSTRALIA